

HYBRID SUBCONVEXITY BOUNDS FOR $L\left(\frac{1}{2}, \text{Sym}^2 f \otimes g\right)$

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ABSTRACT. Fix an integer $\kappa \geq 2$. Let P be prime and let $k > \kappa$ be an even integer. For f a holomorphic cusp form of weight k and full level and g a primitive holomorphic cusp form of weight 2κ and level P , we prove hybrid subconvexity bounds for $L\left(\frac{1}{2}, \text{Sym}^2 f \otimes g\right)$ in the k and P aspects when $P^{\frac{13}{64}+\delta} < k < P^{\frac{3}{8}-\delta}$ for any $0 < \delta < \frac{11}{128}$. These bounds are achieved through a first moment method (with amplification when $P^{\frac{13}{64}} < k \leq P^{\frac{1}{13}}$).

1. INTRODUCTION

Subconvexity estimates for Rankin-Selberg L -functions have been established in a variety of settings recently with strong motivation coming from equidistribution problems of an arithmetic nature. In general, for an L -function $L(s, \pi)$ associated to an irreducible cuspidal automorphic representation π with analytic conductor $Q(s, \pi)$, one hopes to obtain subconvexity estimates of the form $L(s, \pi) \ll Q(s, \pi)^{\frac{1}{4}-\delta}$ for some $\delta > 0$ when $\Re(s) = \frac{1}{2}$. Though the actual value of δ does not often matter in applications, establishing such a subconvexity bound for some $\delta > 0$ is non-trivial and requires careful consideration of the arithmetic/algebraic information associated with π . The convexity bound $L(s, \pi) \ll_{\varepsilon} Q(s, \pi)^{\frac{1}{4}+\varepsilon}$, on the other hand, follows purely from standard tools in complex analysis.

The resolution of one equidistribution problem related to central values of Rankin-Selberg L -functions, the quantum unique ergodicity conjecture of Rudnick and Sarnak [20], has thus far required several techniques from analytic number theory and ergodic theory. In many cases, however, the conjecture would follow directly from subconvexity estimates for $L\left(\frac{1}{2}, \text{Sym}^2 f\right)$ and $L\left(\frac{1}{2}, \text{Sym}^2 f \otimes g\right)$. Here we think of f as a varying modular form and g as a fixed form.

Subconvexity estimates for such L -values have proven to be very difficult to establish through current methods and several authors have first given attention to analogous subconvexity problems for Rankin-Selberg L -functions in order to possibly better understand the structure behind the symmetric square. For a partial list of related works, see [14], [17], [6], [8], [9], [1], [2], [16] and the references therein. For many Rankin-Selberg L -functions, it appears as though the arithmetic/analytic structure of the conductor dictates the method of proof that should be adopted to

achieve subconvexity. For example, an amplified moment method is usually required when only one of the forms in the convolution is varying. However, when at least two of the forms are varying, as in the present work, then a moment computation without amplification suffices in certain hybrid ranges. Curiously, the moment method may be avoided all together in cases where the level of the varying form has special structure, for example if the level of the varying form factorizes in a suitable manner [19].

In the work of Rizwanur Khan [12] on $L\left(\frac{1}{2}, \text{Sym}^2 f \otimes g\right)$, a conditional amplifier of long length relative to the conductor was employed in a first moment method in order to establish subconvexity estimates for fixed f and varying g of prime level P . Following ideas seen in [8] and [9] (among others), the work of Khan [12] suggests that the number of points of summation for the unamplified first moment of $L\left(\frac{1}{2}, \text{Sym}^2 f \otimes g\right)$ is insufficient for application to the subconvexity problem and that one would benefit from increasing the complexity of the L -function by allowing f to vary independently with g .

As we demonstrate in this paper, varying the weight k of f along with the level P of g increases the conductor to be of size $Q\left(\frac{1}{2}, \text{Sym}^2 f \otimes g\right) \asymp k^4 P^3$ and allows us to establish hybrid subconvexity bounds for $L\left(\frac{1}{2}, \text{Sym}^2 f \otimes g\right)$ in the k and P aspects when $P^{\frac{13}{64}+\delta} < k < P^{\frac{3}{8}-\delta}$ for any $0 < \delta < \frac{11}{128}$. Given the above lower bound for k in terms of P , this suggests that much more work remains in establishing subconvexity for the case of P fixed and k varying as required in the holomorphic analogue to the quantum unique ergodicity conjecture. A related situation and hybrid subconvexity bound may be found in ([2], Corollary 1.5) where the authors consider $L\left(\frac{1}{2}, f \otimes g \otimes h\right)$ with all three forms f, g and h varying in weights k, ℓ and $k + \ell$ respectively.

2. STATEMENT OF RESULTS

Fix an integer $\kappa \geq 2$ and newform $g_0 \in H_{2\kappa}^*(P)$ of weight 2κ and level P . Let f be a Hecke eigenform of even weight $k > \kappa$. Let $L \geq 1$ and let \mathcal{P} be a set of primes in the range $[L, 2L]$ not dividing the level P . Our choice for \mathcal{P} will be such that $|\mathcal{P}| \gg \frac{L}{\log L}$. We will be working with an amplified first moment containing g_0

$$\sum_{g \in H_{2\kappa}^*(P)} \omega_g^{-1} |\mathfrak{A}_g|^2 L\left(\frac{1}{2}, \text{Sym}^2 f \otimes g\right)$$

where ω_g is as in (4.15) and the amplifier is given by

$$\mathfrak{A}_g := \sum_{\ell} \alpha_{\ell} \lambda_g(\ell)$$

with

$$(2.1) \quad \alpha_\ell := \begin{cases} \lambda_{g_0}(\ell), & \text{if } \ell \in \mathcal{P}, \\ -1, & \text{if } \ell = p^2 \text{ with } p \in \mathcal{P}, \\ 0, & \text{otherwise.} \end{cases}$$

When $g = g_0$, the Hecke relation $\lambda_{g_0}(p)^2 - \lambda_{g_0}(p^2) = 1$ yields $\mathfrak{A}_{g_0} = |\mathcal{P}|$.

Opening the absolute square and using Hecke multiplicativity gives

$$(2.2) \quad \sum_{\ell_1, \ell_2} \alpha_{\ell_1} \overline{\alpha_{\ell_2}} \sum_{\ell_3 | (\ell_1, \ell_2)} \sum_{g \in H_{2\kappa}^*(P)} \omega_g^{-1} \lambda_g(\ell_1 \ell_2 \ell_3^{-2}) L(\frac{1}{2}, \text{Sym}^2 f \otimes g).$$

In §5, we shall prove the following result.

Theorem 2.1. *Suppose $k > \kappa \geq 2$ are integers, with k even, P is a prime, and f is a Hecke cusp form of weight k for $\text{SL}(2, \mathbb{Z})$. Let $\ell \leq 16L^4$ be a positive integer. Then under the assumption*

$$(2.3) \quad L \leq k^{-\frac{2}{5}} P^{\frac{3}{20}},$$

we have for any $\varepsilon > 0$

$$\sum_{g \in H_{2\kappa}^*(P)} \omega_g^{-1} \lambda_g(\ell) L(\frac{1}{2}, \text{Sym}^2 f \otimes g) \ll_{\varepsilon, \kappa} \left(\frac{1}{\sqrt{\ell}} + \frac{L^{\frac{22}{7}} k^{\frac{13}{7}}}{P^{\frac{4}{7}}} \right) (kP)^\varepsilon.$$

Remark 2.2. (1) *The assumptions $\kappa \geq 2$ and (2.3) are a result of technical difficulties in the proof. See Remark 4.1 and 5.1.*

(2) *Setting $\ell = L = 1$, we note that the above bound is the Lindelöf on average bound when $k \leq P^{\frac{4}{13}}$. Therefore, this is the only range in which amplification is applied.*

Inserting the above bound into (2.2) and trivially averaging over ℓ_1 and ℓ_2 we get

$$\sum_{g \in H_{2\kappa}^*(P)} \omega_g^{-1} |\mathfrak{A}_g|^2 L(\frac{1}{2}, \text{Sym}^2 f \otimes g) \ll_{\varepsilon, \kappa} \left(L + \frac{L^{\frac{36}{7}} k^{\frac{13}{7}}}{P^{\frac{4}{7}}} \right) (kP)^\varepsilon.$$

Using the non-negativity of the central L -values and the definition of our amplifier according to (2.1) gives

$$L(\frac{1}{2}, \text{Sym}^2 f \otimes g_0) \ll_{\varepsilon, \kappa} \left(\frac{P}{L} + L^{\frac{22}{7}} k^{\frac{13}{7}} P^{\frac{3}{7}} \right) (kP)^\varepsilon.$$

Finally, setting

$$(2.4) \quad L = \begin{cases} k^{-\frac{13}{29}} P^{\frac{4}{29}}, & \text{if } P^{\frac{13}{64}} < k \leq P^{\frac{4}{13}}, \\ 1, & \text{if } P^{\frac{4}{13}} < k < P^{\frac{3}{8}}, \end{cases}$$

one verifies that the assumption (2.3) on L is satisfied and we therefore obtain the following corollary.

Corollary 2.3. *For f as above and g a newform of weight $2k$ and level P , we have*

$$(2.5) \quad L\left(\frac{1}{2}, \text{Sym}^2 f \otimes g\right) \ll_{\varepsilon, \kappa} \begin{cases} k^{\frac{13}{29}} P^{\frac{25}{29}} (kP)^{\varepsilon}, & \text{if } P^{\frac{13}{64}} < k \leq P^{\frac{4}{13}}, \\ \left(P + k^{\frac{13}{7}} P^{\frac{3}{7}}\right) (kP)^{\varepsilon}, & \text{if } P^{\frac{4}{13}} < k < P^{\frac{3}{8}}. \end{cases}$$

Remark 2.4. *Note that (2.5) beats the convexity bound $kP^{\frac{3}{4}}(kP)^{\varepsilon}$ when $P^{\frac{13}{64}+\delta} < k < P^{\frac{3}{8}-\delta}$ for some $0 < \delta < \frac{11}{128}$. Putting $\ell = L = 1$ in Theorem 2.1, we arrive at the following bound by non-negativity*

$$L\left(\frac{1}{2}, \text{Sym}^2 f \otimes g\right) \ll_{\varepsilon, \kappa} \left(P + k^{\frac{13}{7}} P^{\frac{3}{7}}\right) (kP)^{\varepsilon},$$

which is extracted from the second line of (2.5). This bound is already able to beat the convexity bound when $P^{\frac{1}{4}+\delta} < k < P^{\frac{3}{8}-\delta}$, and therefore amplification is unnecessary (although the bound from amplification, i.e. the first line of (2.5), also provides a subconvexity bound on the overlapping range $P^{\frac{1}{4}} < k \leq P^{\frac{4}{13}}$). Thus, the amplification method extends the range of admissible exponents from below by $\frac{3}{64}$.

3. SKETCH OF HYBRID SUBCONVEXITY IN A SIMPLIFIED CASE

Let f be a holomorphic cusp form of even weight k and full level and let g be a primitive holomorphic cusp form of even weight 2κ and prime level P . In order to demonstrate the ideas behind the proofs of our main results, we provide a brief sketch of how one might establish hybrid subconvexity bounds when κ is large and fixed. For notational convenience, we denote the Dirichlet coefficients of $\text{Sym}^2 f$ by $A(n)$ and the coefficients of g by $\lambda(n)$ such that a standard approximate functional equation argument will essentially equate our central L -value $L\left(\frac{1}{2}, \text{Sym}^2 f \otimes g\right)$ with

$$D_g^1(Y) + D_g^2(Y) := \sum_{n \ll Y\sqrt{Q}} \frac{A(n)\lambda(n)}{\sqrt{n}} + \varepsilon(\text{Sym}^2 f \otimes g) \sum_{n \ll \sqrt{Q}/Y} \frac{A(n)\lambda(n)}{\sqrt{n}}$$

for any $Y > 0$ with $Q = k^4 P^3 \asymp Q\left(\frac{1}{2}, \text{Sym}^2 f \times g\right)$ and root number $\varepsilon(\text{Sym}^2 f \otimes g) = (-1)^{\kappa} \sqrt{P} \lambda(P) = \pm 1$.

Our method will be a normalized first moment average over newforms g . Therefore, we wish to achieve a better result than the first moment convexity bound

$$(3.1) \quad \sum_{g \in H_{2\kappa}^*(P)} \omega_g^{-1} L\left(\frac{1}{2}, \text{Sym}^2 f \otimes g\right) \ll \frac{Q^{\frac{1}{4}+\varepsilon}}{P}$$

where ω_g is as in (4.15). As noted in the previous section, one gains from amplification in certain ranges of k relative to P , but we omit this component here.

Assume that for our particular choice of κ and P , the space of newforms $H_{2\kappa}^*(P)$ spans the space of all forms $S_{2\kappa}(P)$. Write

$$S_1(Y) + S_2(Y) := \sum_{g \in H_{2\kappa}^*(P)} \omega_g^{-1} D_g^1(Y) + \sum_{g \in H_{2\kappa}^*(P)} \omega_g^{-1} D_g^2(Y)$$

and consider first $S_1(Y)$. Applying the Petersson trace formula in the average over g along with standard Bessel function bounds (4.21) and the Weil bound for Kloosterman sums, one obtains

$$\begin{aligned} S_1(Y) &= 1 + 2\pi(-1)^\kappa \sum_{\substack{c \equiv 0 \pmod{P} \\ c > 0}} \frac{1}{c} \sum_{n \ll Y\sqrt{Q}} \frac{A(n)}{\sqrt{n}} S(n, 1; c) J_{2\kappa-1}\left(\frac{4\pi\sqrt{n}}{c}\right) \\ &\ll_\kappa 1 + \sum_{\substack{c \equiv 0 \pmod{P} \\ c > 0}} \frac{1}{\sqrt{c}} \sum_{n \ll Y\sqrt{Q}} \frac{|A(n)|}{\sqrt{n}} \left(\frac{\sqrt{n}}{c}\right)^{2\kappa-1}. \end{aligned}$$

The Deligne bound for the coefficients of holomorphic forms then gives

$$(3.2) \quad S_1(Y) \ll_{\varepsilon, \kappa} \max \left\{ 1, \sqrt{P} \left(\frac{Y\sqrt{Q}}{P^2} \right)^\kappa \right\} Q^\varepsilon.$$

Note that the condition $1 < \frac{Q^{\frac{1}{4}}}{P}$ or equivalently $P^{\frac{1}{4}} < k$ is necessary for the bound (3.2) above to be better than the first moment convexity bound (3.1).

Remark 3.1. *We shall see that such basic analysis of $S_1(Y)$, when κ is large, is sufficient for establishing at least some hybrid subconvexity bound due to the congruence condition in the sum over c . In §5, we improve on the above bound (3.2) in order to establish our main results in §2.*

Now for $S_2(Y)$, an application of Petersson's trace formula shows that $S_2(Y)$ is essentially equal to

$$\begin{aligned}
& \sum_{\substack{c \equiv 0 \pmod{P} \\ c > 0}} \frac{\sqrt{P}}{c} \sum_{n \ll \sqrt{Q}/Y} \frac{A(n)}{\sqrt{n}} S(nP, 1; c) J_{2\kappa-1} \left(\frac{4\pi \sqrt{nP}}{c} \right) \\
&= \sum_{\substack{(c,P)=1 \\ c > 0}} \frac{1}{c \sqrt{P}} \sum_{n \ll \sqrt{Q}/Y} \frac{A(n)}{\sqrt{n}} S(nP, 1; cP) J_{2\kappa-1} \left(\frac{4\pi \sqrt{n}}{c \sqrt{P}} \right) \\
&= - \sum_{\substack{(c,P)=1 \\ c > 0}} \frac{1}{c \sqrt{P}} \sum_{n \ll \sqrt{Q}/Y} \frac{A(n)}{\sqrt{n}} S(n, \bar{P}; c) J_{2\kappa-1} \left(\frac{4\pi \sqrt{n}}{c \sqrt{P}} \right).
\end{aligned}$$

Here we pulled out the P divisor in the original c -sum and used basic properties of the Kloosterman sums, i.e.

$$S(nP, 1; cP) = \begin{cases} 0, & \text{if } P|c, \\ S(0, \bar{c}; P) S(n, \bar{P}; c) = -S(n, \bar{P}; c), & \text{if } (c, P) = 1. \end{cases}$$

Focusing on the transition range of the Bessel function ($\sqrt{n} \asymp c \sqrt{P}$) for the remainder of the sketch and opening the Kloosterman sum, we see that we must analyze a smoothed version of

$$\sum_{\substack{(c,P)=1 \\ c \asymp Q^{\frac{1}{4}}/\sqrt{YP}}} \frac{1}{c \sqrt{P}} \sum_{a \pmod{c}}^* e\left(\frac{a\bar{P}}{c}\right) \sum_{n \ll \sqrt{Q}/Y} \frac{A(n)}{\sqrt{n}} e\left(\frac{n\bar{a}}{c}\right).$$

An application of Voronoï summation in n leads to sums of the form

$$\sum_{\substack{(c,P)=1 \\ c \asymp Q^{\frac{1}{4}}/\sqrt{YP}}} \frac{1}{c \sqrt{P}} \sum_{a \pmod{c}}^* e\left(\frac{a\bar{P}}{c}\right) \sum_{n \ll \frac{c^3 k^2}{\sqrt{Q}/Y}} \frac{A(n)}{\sqrt{n}} \frac{S(a, n; c)}{\sqrt{c}}$$

i.e. we obtain a new n -sum of length “conductor divided by the original length of summation” with a summand “dual” to the previous summand. Summing over a , one sees that we must consider

$$\sum_{\substack{(c,P)=1 \\ c \asymp Q^{\frac{1}{4}}/\sqrt{YP}}} \frac{1}{\sqrt{cP}} \sum_{n \ll \frac{k^2 Q^{\frac{1}{4}}}{\sqrt{YP^3}}} \frac{A(n)}{\sqrt{n}} e\left(-\frac{nP}{c}\right).$$

Trivially bounding the sum over n using Deligne's bound, the above is bounded by

$$(3.3) \quad \sum_{\substack{(c,P)=1 \\ c \asymp Q^{\frac{1}{4}}/\sqrt{YP}}} \frac{1}{\sqrt{cP}} \left(\frac{kQ^{\frac{1}{8}}}{Y^{\frac{1}{4}}P^{\frac{3}{4}}} \right)^{1+\varepsilon} \ll \frac{Q^{\frac{1}{4}+\varepsilon}}{P} \left(\frac{k}{\sqrt{YP}} \right).$$

This bound is better than the first moment convexity bound (3.1) when $Y > \frac{k^2}{P}$.

We now combine the bounds in (3.2) and (3.3). Assume first that

$$\sqrt{P} \left(\frac{Y\sqrt{Q}}{P^2} \right)^\kappa \leq 1 < \frac{Q^{\frac{1}{4}}}{P}.$$

Equating the two bounds $1 = \frac{Q^{\frac{1}{4}}}{P} \left(\frac{k}{\sqrt{YP}} \right)$ we get $Y = \frac{k^2\sqrt{Q}}{P^3} = k^4P^{-\frac{3}{2}}$. Such a choice of Y satisfies our assumption when $P^{\frac{1}{4}} < k \leq P^{\frac{1}{3}-\frac{1}{12\kappa}}$. Now assume that

$$1 \leq \sqrt{P} \left(\frac{Y\sqrt{Q}}{P^2} \right)^\kappa < \frac{Q^{\frac{1}{4}}}{P}.$$

Equating the two bounds $\sqrt{P} \left(\frac{Y\sqrt{Q}}{P^2} \right)^\kappa = \frac{Q^{\frac{1}{4}}}{P} \left(\frac{k}{\sqrt{YP}} \right)$ we get $Y = k^{\frac{4-4\kappa}{2\kappa+1}} P^{\frac{2\kappa-5}{4\kappa+2}}$.

Such a choice of Y satisfies our assumption when $P^{\frac{1}{3}-\frac{1}{12\kappa}} \leq k < P^{\frac{3(2\kappa-1)}{4(4\kappa-1)}}$.

Therefore, one establishes hybrid subconvexity bounds for all $\kappa \geq 2$ with the range of k relative to P tending to $P^{\frac{1}{4}} < k < P^{\frac{3}{8}}$ as $\kappa \rightarrow \infty$.

4. PRELIMINARIES

4.1. Holomorphic cusp forms. For a positive integer N and an even positive integer k , the space $S_k(N)$ of cusp forms of weight k for the Hecke congruence group $\Gamma_0(N)$ is a finite-dimensional Hilbert space with respect to the Petersson inner product

$$\langle f_1, f_2 \rangle := \int_{\Gamma_0(N) \backslash \mathbb{H}} f_1(z) \overline{f_2(z)} y^{k-2} dx dy, \quad f_1, f_2 \in S_k(N),$$

where \mathbb{H} denotes the upper half-plane. Every $f \in S_k(N)$ has a Fourier series expansion

$$f(z) = \sum_{n=1}^{\infty} \psi_f(n) n^{\frac{k-1}{2}} e(nz).$$

For $n \geq 1$, define the Hecke operator $T_N(n)$ by

$$(T_N(n)f)(z) := \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ (a,N)=1}} \left(\frac{a}{d} \right)^{\frac{k}{2}} \sum_{b \pmod{d}} f\left(\frac{az+b}{d} \right).$$

Let $H_k^*(N)$ be the orthogonal set of Hecke-normalized (i.e., $\psi_f(1) = 1$) newforms f in $S_k(N)$. Every $f \in H_k^*(N)$ is an eigenfunction of all Hecke operators

$T_N(n)$; let $\lambda_f(n)$ be its eigenvalue of $T_N(n)$. We have $\psi_f(n) = \lambda_f(n)$ for all $n \geq 1$. The Hecke eigenvalues are multiplicative, i.e., for any $m, n \geq 1$

$$(4.1) \quad \lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n) \\ (d,N)=1}} \lambda_f(mn/d^2).$$

In particular, (4.1) becomes completely multiplicative when $n|N$

$$(4.2) \quad \lambda_f(m)\lambda_f(n) = \lambda_f(mn).$$

For any $f \in H_k^*(N)$, we have Deligne's bound

$$|\lambda_f(n)| \leq \tau(n),$$

and when N is squarefree, it is known that ([11, (2.24)])

$$\lambda_f(n)^2 = \frac{1}{n}, \quad \text{if } n|N.$$

4.2. Automorphic L -functions. In this section some preliminary results on automorphic L -functions are given. We shall particularly focus on the Rankin-Selberg L -function $L(s, \text{Sym}^2 f \otimes g)$ for $f \in S_k(1)$ and $g \in H_{2\kappa}^*(N)$ with N squarefree, k an even positive integer and κ a positive integer. A brief calculation of the γ -factor and the ε -factor of $\text{Sym}^2 f \otimes g$ will be given in §4.2.4.

4.2.1. For $f \in H_k^*(N)$ the Hecke L -function is defined by

$$L(s, f) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s}.$$

This has an Euler product $L(s, f) = \prod_p L_p(s, f)$ with local factors

$$L_p(s, f) = (1 - \lambda_f(p)p^{-s} + \chi_0(p)^{-2s})^{-1},$$

where χ_0 is the principal character modulus N . The gamma factor is

$$\gamma(s, f) = \pi^{-s} \Gamma\left(\frac{s}{2} + \frac{k-1}{4}\right) \Gamma\left(\frac{s}{2} + \frac{k+1}{4}\right).$$

The complete product $\Lambda(s, f) = N^{\frac{s}{2}} \gamma(s, f) L(s, f)$ is entire and satisfies the functional equation

$$\Lambda(s, f) = \varepsilon_f \Lambda(1-s, f),$$

with root number $\varepsilon_f = i^k \eta_f = \pm 1$, where η_f is the eigenvalue of the Atkin-Lehner involution W_N . If N is squarefree, then $\varepsilon(f) = i^k \mu(N) \lambda_f(N) \sqrt{N}$.

For $p \nmid N$, the local factors $L_p(s, f)$ factor further as

$$L_p(s, f) = (1 - \alpha_f(p)p^{-s})^{-1} (1 - \beta_f(p)p^{-s})^{-1},$$

where $\alpha_f(p)$, $\beta_f(p)$ are complex numbers with $\alpha_f(p) + \beta_f(p) = \lambda_f(p)$ and $\alpha_f(p)\beta_f(p) = 1$.

4.2.2. For $f \in S_k(1)$ the symmetric square L -function is defined by

$$L(s, \text{Sym}^2 f) = \zeta(2s)Z(s, f),$$

where $\zeta(s)$ denotes the Riemann zeta function and $Z(s, f)$ is defined by

$$Z(s, f) = \sum_{n=1}^{\infty} \lambda_f(n^2) n^{-s}.$$

This has Euler product $L(s, \text{Sym}^2 f) = \prod_p L_p(s, \text{Sym}^2 f)$ with

$$L_p(s, \text{Sym}^2 f) = (1 - \alpha_f^2(p)p^{-s})^{-1}(1 - p^{-s})^{-1}(1 - \beta_f^2(p)p^{-s})^{-1}.$$

The gamma factor is

$$\gamma(s, \text{Sym}^2 f) = \pi^{-\frac{3s}{2}} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s+k-1}{2}\right) \Gamma\left(\frac{s+k}{2}\right)$$

The complete product $\Lambda(s, \text{Sym}^2 f) = \gamma(s, \text{Sym}^2 f)L(s, \text{Sym}^2 f)$ is entire and it satisfies the functional equation

$$\Lambda(s, \text{Sym}^2 f) = \Lambda(1-s, \text{Sym}^2 f).$$

We have the convexity bound

$$(4.3) \quad L(\sigma + it, \text{Sym}^2 f) \ll_{\varepsilon} ((|t| + 1)(|t| + k)^2)^{\frac{1-\sigma}{2} + \varepsilon}, \quad 0 \leq \sigma \leq 1,$$

where the implied constant depends only on $\varepsilon > 0$. Moreover, it is known that [7]

$$(4.4) \quad k^{-\varepsilon} \ll_{\varepsilon} L(1, \text{Sym}^2 f) \ll_{\varepsilon} k^{\varepsilon}.$$

According to [3], $L(s, \text{Sym}^2 f)$ is also an L -function $L(s, F)$ of some automorphic representation F of $\text{GL}(3, \mathbb{Z})$, and the normalized Fourier coefficients are given by

$$\begin{aligned} A_F(1, n) &= A_F(n, 1) = \sum_{m \ell^2 = n} \lambda_f(m^2), \\ A_F(m, n) &= A_F(-m, n) = A_F(m, -n) = A_F(-m, -n), \\ A_F(m, 0) &= A_F(0, n) = 0, \end{aligned}$$

and the Hecke relations ([18, (7.7)])

$$(4.5) \quad A_F(m, n) = \sum_{d|(m, n)} \mu(d) A_F(m/d, 1) A_F(1, n/d).$$

We have

$$L(s, \text{Sym}^2 f) = L(s, F) = \sum_{n=1}^{\infty} A_F(1, n) n^{-s}.$$

4.2.3. Let $f \in S_k(1)$ and $g \in H_{2\kappa}^*(N)$ with N squarefree, k an even positive integer and κ a positive integer. We define the Rankin-Selberg L -function

$$L(s, \text{Sym}^2 f \otimes g) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_F(m, n) \lambda_g(n) (m^2 n)^{-s}.$$

This has Euler product $L(s, \text{Sym}^2 f \otimes g) = \prod_p L_p(s, \text{Sym}^2 f \otimes g)$ with local factor

$$\begin{aligned} & L_p(s, \text{Sym}^2 f \otimes g) \\ &= (1 - \alpha_f^2(p) \alpha_g(p) p^{-s})^{-1} (1 - \alpha_g(p) p^{-s})^{-1} (1 - \beta_f^2(p) \alpha_g(p) p^{-s})^{-1} \\ & \quad (1 - \alpha_f^2(p) \beta_g(p) p^{-s})^{-1} (1 - \beta_g(p) p^{-s})^{-1} (1 - \beta_f^2(p) \beta_g(p) p^{-s})^{-1} \end{aligned}$$

if $p \nmid N$, and

$$\begin{aligned} & L_p(s, \text{Sym}^2 f \otimes g) \\ &= (1 - \alpha_f^2(p) \lambda_g(p) p^{-s})^{-1} (1 - \lambda_g(p) p^{-s})^{-1} (1 - \beta_f^2(p) \lambda_g(p) p^{-s})^{-1} \end{aligned}$$

if $p|N$. The gamma factor is

$$\begin{aligned} & \gamma(s, \text{Sym}^2 f \otimes g) \\ (4.6) \quad &= (2\pi)^{-3s} \Gamma\left(s + \kappa - \frac{1}{2}\right) \Gamma\left(s + k + \kappa - \frac{3}{2}\right) \Gamma\left(s + \left|k - \kappa - \frac{1}{2}\right|\right). \end{aligned}$$

The complete product $\Lambda(s, \text{Sym}^2 f \otimes g) = N^{\frac{3s}{2}} \gamma(s, \text{Sym}^2 f \otimes g) L(s, \text{Sym}^2 f \otimes g)$ is entire and it satisfies the functional equation

$$\Lambda(s, \text{Sym}^2 f \otimes g) = \varepsilon(\text{Sym}^2 f \otimes g) \Lambda(1 - s, \text{Sym}^2 f \otimes g),$$

with root number $\varepsilon(\text{Sym}^2 f \otimes g) = \pm 1$ given by

$$(4.7) \quad \varepsilon(\text{Sym}^2 f \otimes g) = \begin{cases} (-1)^{\kappa+1} \mu(N) \lambda_g(N) \sqrt{N}, & \text{if } k > \kappa, \\ (-1)^{\kappa} \mu(N) \lambda_g(N) \sqrt{N}, & \text{if } k \leq \kappa. \end{cases}$$

It follows from the Hecke relations (4.5) and (4.2) that

$$\begin{aligned} & L(s, \text{Sym}^2 f \otimes g) \\ &= \sum_{d=1}^{\infty} \mu(d) d^{-3s} \sum_{m=1}^{\infty} A_F(m, 1) m^{-2s} \sum_{n=1}^{\infty} A_F(1, n) \lambda_g(dn) n^{-s} \\ (4.8) \quad &= L(2s, \text{Sym}^2 f) \sum_{d=1}^{\infty} \mu(d) d^{-3s} \sum_{n=1}^{\infty} A_F(1, n) \lambda_g(dn) n^{-s} \\ &= L(2s, \text{Sym}^2 f) L_N(3s, g)^{-1} \sum_{(d, N)=1} \mu(d) d^{-3s} \sum_{n=1}^{\infty} A_F(1, n) \lambda_g(dn) n^{-s}, \end{aligned}$$

where $L_N(s, g)$ denotes the finite Euler product

$$L_N(s, g) = \prod_{p|N} (1 - \lambda_g(p)p^{-s})^{-1}.$$

4.2.4. *Computation of the γ -factor and the ε -factor of $\text{Sym}^2 f \otimes g$.* Let ψ_∞ be the standard additive character on \mathbb{R} , namely $\psi_\infty(x) = e(x)$, and ψ_p be a normalized unramified additive character of \mathbb{Q}_p for each prime p .

At the real place, the local component $\pi_{f,\infty}$, respectively $\pi_{g,\infty}$, is the discrete series representation of $\text{GL}(2, \mathbb{R})$ with weight k , respectively 2κ .

The Weil group The Weil group $W_{\mathbb{R}}$ of \mathbb{R} is realized as $\mathbb{C}^\times \cup j\mathbb{C}^\times$ satisfying $j^2 = -1 \in \mathbb{C}^\times$ and $jzj^{-1} = \bar{z}$ for $z \in \mathbb{C}^\times$. Under the local Langlands correspondence the discrete series with weight k , $k \geq 2$, corresponds to the two dimensional representation ρ_k of $W_{\mathbb{R}}$ given by

$$\rho_k(re^{i\theta}) = \begin{pmatrix} e^{i(k-1)\theta} & \\ & e^{-i(k-1)\theta} \end{pmatrix} \quad \rho_k(j) = \begin{pmatrix} & (-1)^{k-1} \\ 1 & \end{pmatrix}$$

The γ -factor $\gamma(s, \rho_k) = \Gamma_{\mathbb{C}}(s + \frac{k-1}{2})$ and the ε -factor $\varepsilon(\rho_k, \psi_\infty) = i^k$. See [13]. With some matrix calculations we have

$$\text{Sym}^2(\rho_k) \otimes \rho_{2\kappa} = \rho_{2\kappa} \oplus \rho_{|2\kappa-2k-1|+1} \oplus \rho_{2\kappa+2k-2}.$$

This implies the formula (4.6) for the γ -factor $\gamma(s, \text{Sym}^2 f \otimes g)$, and the ε -factor at ∞ is

$$(4.9) \quad \varepsilon_\infty(\text{Sym}^2 f \otimes g, \psi_\infty) = \begin{cases} (-1)^{\kappa+1}, & \text{if } k > \kappa, \\ (-1)^\kappa, & \text{if } k \leq \kappa. \end{cases}$$

For any prime p , the local component $\pi_{\text{Sym}^2 f, p}$ is an unramified principle series representation of $\text{GL}(3, \mathbb{Q}_p)$ with trivial central character, so

$$(4.10) \quad \varepsilon_p(\text{Sym}^2 f \otimes g, \psi_p) = \varepsilon_p(g, \psi_p)^3 = \begin{cases} -\lambda_g(p) \sqrt{p}, & \text{if } p|N, \\ 1, & \text{if } p \nmid N. \end{cases}$$

Multiplying (4.9) and (4.10) yields the formula (4.7) for the ε -factor $\varepsilon(\text{Sym}^2 f \otimes g)$.

4.3. Approximate functional equation. In view of (4.8), we have the following approximate functional equation (see [10, Theorem 5.3, Proposition 5.4])

$$\begin{aligned}
 & L\left(\frac{1}{2}, \text{Sym}^2 f \otimes g\right) \\
 (4.11) \quad &= \sum_{(d,N)=1} \sum_{n=1}^{\infty} \frac{\mu(d) A_F(1, n) \lambda_g(dn)}{\sqrt{d^3 n}} V\left(\frac{d^3 n}{YN^{\frac{3}{2}}}\right) \\
 &+ \varepsilon(\text{Sym}^2 f \otimes g) \sum_{(d,N)=1} \sum_{n=1}^{\infty} \frac{\mu(d) A_F(1, n) \lambda_g(dn)}{\sqrt{d^3 n}} V\left(\frac{d^3 n Y}{N^{\frac{3}{2}}}\right),
 \end{aligned}$$

where $V(y)$ is a smooth function on \mathbb{R}_+ defined by

$$\frac{1}{2\pi i} \int_{(3)} \left(\cos\left(\frac{\pi u}{4A}\right) \right)^{-24A} \frac{\gamma\left(\frac{1}{2} + u, \text{Sym}^2 f \otimes g\right)}{\gamma\left(\frac{1}{2}, \text{Sym}^2 f \otimes g\right)} \frac{L(1 + 2u, \text{Sym}^2 f)}{L_N\left(\frac{3}{2} + 3u, g\right)} y^{-u} \frac{du}{u},$$

with A a positive integer. For y large we shift the contour of integration in $V(y)$ to $\Re u = A$, and for y small we left shift the contour to $\Re u = -\sigma$ with $0 < \sigma < \frac{1}{2}$, passing through the pole $u = 0$ with residue $L(1, \text{Sym}^2 f) L_N\left(\frac{3}{2}, g\right)^{-1}$. Then by Stirling's formula and the convexity bound (4.3) for $L(s, \text{Sym}^2 f)$ we derive

$$(4.12) \quad y^j V^{(j)}(y) \ll_{\varepsilon, j, A, \kappa} (kN)^{\varepsilon} \left(1 + \frac{y}{k^2}\right)^{-A},$$

and the asymptotic equation

$$(4.13) \quad y^j V^{(j)}(y) = L(1, \text{Sym}^2 f) L_N\left(\frac{3}{2}, g\right)^{-1} \delta(0, j) + O_{\varepsilon, \sigma, j, \kappa}((kN)^{\varepsilon} y^{\sigma}).$$

Choosing σ sufficiently small, we have on the range $y < k^{2+\varepsilon}$

$$(4.14) \quad y^j V^{(j)}(y) \ll_{\varepsilon, j, \kappa} (kN)^{\varepsilon}.$$

4.4. Petersson trace formula. Let $\mathcal{B}_k(N)$ be an orthogonal basis of $S_k(N)$. For any $n, m \geq 1$ define

$$\Delta_{k,N}(m, n) := \sum_{f \in \mathcal{B}_k(N)} \omega_f^{-1} \psi_f(m) \psi_f(n).$$

This is basis independent. Here the weight ω_f is defined by

$$(4.15) \quad \omega_f := \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \langle f, f \rangle.$$

If $f \in H_k^{\star}(N)$, then

$$\omega_f = \frac{(k-1)N}{2\pi^2} L(1, \text{Sym}^2 f),$$

and by (4.4) we have

$$(4.16) \quad (kN)^{1-\varepsilon} \ll_{\varepsilon} \omega_f \ll_{\varepsilon} (kN)^{1+\varepsilon}.$$

Moreover, it follows from [11, (2.48, 2.72)], the bound (4.4) of $L(1, \text{Sym}^2 f)$ and Deligne's bound that

$$(4.17) \quad \Delta_{k,N}(m, n) \ll_{\varepsilon} (Nknm)^{\varepsilon}.$$

We have the following formula of Petersson.

$$(4.18) \quad \Delta_{k,N}(m, n) = \delta(m, n) + 2\pi i^{-k} \sum_{\substack{c>0 \\ c \equiv 0 \pmod{N}}} \frac{S(m, n; c)}{c} J_{k-1} \left(\frac{4\pi \sqrt{mn}}{c} \right),$$

where J_{k-1} is the Bessel function of the first kind with order $k-1$.

Define

$$\Delta_{k,N}^*(m, n) := \sum_{f \in H_k^*(N)} \omega_f^{-1} \lambda_f(m) \lambda_f(n).$$

According to [11, Proposition 2.8], under the assumptions that N be squarefree, $(m, N) = 1$ and $(n, N) | N^2$ we have

$$(4.19) \quad \Delta_{k,N}^*(m, n) = \sum_{LR=N} \frac{\mu(L)}{L\nu((n, L))} \sum_{\ell | L^{\infty}} \ell^{-1} \Delta_{k,R}(m\ell^2, n),$$

with

$$\nu(n) = n \prod_{p|n} \left(1 + \frac{1}{p} \right).$$

For our purpose, a more general variant of (4.19) given in [9, Lemma 2.4] is more convenient. Suppose N is squarefree and $(m, N) = 1$, then

$$(4.20) \quad \Delta_{k,N}^*(m, n) = \sum_{LR=N} \frac{\mu(L)}{L\nu((n, L))} \sum_{\ell | L^{\infty}} \ell^{-1} \sum_{\ell_1^2 | (n, \ell_1 L)} \mu(\ell_1) \ell_1 \Delta_{k,R}(m\ell^2, n\ell_1^{-2}).$$

4.5. Bessel functions. Let ν be a positive integer. If $x \ll 1$, the Taylor series expansion yields

$$(4.21) \quad x^j J_{\nu}^{(j)}(x) \ll_{\nu, j} x^{\nu}, \quad j \geq 0.$$

We have (see [22, §16.12, §17.5])

$$(4.22) \quad J_{\nu}(x) = \frac{1}{\sqrt{2\pi x}} (e^{ix} W_{\nu,+}(x) + e^{-ix} W_{\nu,-}(x)),$$

where

$$\begin{aligned} W_{\nu,\pm}(x) &= e^{\mp i(x + \frac{1}{2}(\nu + \frac{1}{2})\pi i)} W_{0,\nu}(\mp 2ix) \\ &= \frac{e^{\mp \frac{1}{2}(\nu + \frac{1}{2})\pi i}}{\Gamma(\nu + \frac{1}{2})} \int_0^{\infty} e^{-y} \left(y \left(1 \pm i \frac{y}{2x} \right) \right)^{\nu - \frac{1}{2}} dy. \end{aligned}$$

For $x \gg 1$, the asymptotic expansion for the Whittaker functions ([22, §16.3] or [5, 9.227]) and their recursion formula ([5, 9.234.3]) provide the bound

$$(4.23) \quad x^j W_{\nu, \pm}^{(j)}(x) \ll_{\nu, j} 1.$$

Remark 4.1. *The authors would like to point out a common mistake presented in literatures. (4.23) does not hold for small $x \ll 1$. Actually, we have*

$$J_\nu(x) + iY_\nu(x) = H_\nu^{(1)}(x) = \sqrt{\frac{2}{\pi x}} e^{ix} W_{\nu, +}(x),$$

where Y_ν is the Bessel function of the second kind and $H_\nu^{(1)}$ is the Hankel function. The behaviour of Y_ν near zero is given by [21, 3.52 (3)] as follows

$$\begin{aligned} Y_\nu(x) = & -\frac{1}{\pi} \sum_{n=0}^{\nu-1} \frac{(\nu-n-1)!}{n!} \left(\frac{1}{2}x\right)^{-\nu-2n} \\ & + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}x\right)^{\nu+2n}}{n!(\nu+n)!} \left(2 \log\left(\frac{1}{2}x\right) - \psi(n+1) - \psi(\nu+n+1)\right), \end{aligned}$$

with $\psi(1) = -\gamma$, $\psi(n+1) = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \gamma$, where γ denotes Euler's constant. In particular, $W_{\nu, +}(x)$ tends to infinity with growth rate $x^{\frac{1}{2}-\nu}$ as x tends to zero.

Thus, for small x , one has to apply the bound (4.21) of $J_\nu(x)$, as done for instance in §4.8.1. However, $\nu \geq 3$, or $\kappa \geq 2$ in our context, is required to guarantee convergence of certain series as shown in (5.12, 5.16).

4.6. Voronoï formula. For a smooth compactly-supported function $\psi(y)$ on \mathbb{R}_+ , define the Mellin transform by

$$\tilde{\psi}(s) := \int_0^\infty \psi(y) y^s \frac{dy}{y}.$$

Let $\eta \in \{0, 1\}$. For $\sigma > -1$ define

$$\Psi_\eta(x) := \frac{1}{2\pi i} \int_{(\sigma)} (\pi^3 x)^{-s} G_\eta(s) \tilde{\psi}(-s) ds,$$

where

$$(4.24) \quad G_\eta(s) := \frac{\Gamma\left(\frac{1+s+1-\eta}{2}\right) \Gamma\left(\frac{1+s+(k-1)+\eta}{2}\right) \Gamma\left(\frac{1+s+k-\eta}{2}\right)}{\Gamma\left(\frac{-s+1-\eta}{2}\right) \Gamma\left(\frac{-s+(k-1)+\eta}{2}\right) \Gamma\left(\frac{-s+k-\eta}{2}\right)}.$$

Then define

$$\begin{aligned} \Psi_+(x) &:= \frac{1}{2\pi^{\frac{3}{2}}} (\Psi_0(x) - i\Psi_1(x)), \\ \Psi_-(x) &:= \frac{1}{2\pi^{\frac{3}{2}}} (\Psi_0(x) + i\Psi_1(x)). \end{aligned}$$

We have the following Voronoï formula ([18, Theorem 1.18]).

Proposition 4.2. *Let $\psi(y)$ be a smooth function compactly supported on \mathbb{R}_+ . Let $a, \bar{a}, c \in \mathbb{Z}$ with $c \neq 0$, $(a, c) = 1$ and $a\bar{a} \equiv 1 \pmod{c}$. Then we have*

$$\begin{aligned} \sum_{n=1}^{\infty} A_F(1, n) e\left(\frac{n\bar{a}}{c}\right) \psi(n) = & c \sum_{n_1|c} \sum_{n_2=1}^{\infty} \frac{A_F(n_2, n_1)}{n_1 n_2} S(a, n_2; c/n_1) \Psi_+\left(\frac{n_2 n_1^2}{c^3}\right) \\ & + c \sum_{n_1|c} \sum_{n_2=1}^{\infty} \frac{A_F(n_2, n_1)}{n_1 n_2} S(a, -n_2; c/n_1) \Psi_-\left(\frac{n_2 n_1^2}{c^3}\right). \end{aligned}$$

Remark 4.3. *We have used the functional equation*

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}$$

and the fact that k is even to rewrite the Gamma factors in [18, Theorem 1.18] into the form in (4.24).

4.7. Gamma function. Fix $s_0 \in \mathbb{C}$ and let $\Re s > -\Re s_0$. We have an asymptotic expansion as $|\Im s| \rightarrow \infty$

$$\log \Gamma(s_0 + s) = \left(s_0 + s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log(2\pi) + O\left(\frac{1}{|s|}\right).$$

With some calculations we show

$$(4.25) \quad |G_\eta(\sigma + it)| \ll ((|t| + 1)(t^2 + k^2))^{\sigma + \frac{1}{2}}.$$

4.8. Integral transform. Let $X > 0$. Suppose function $w(y)$ satisfies

$$(4.26) \quad \begin{cases} w(y) \text{ is smooth with support in the dyadic interval } [X, 2X], \\ y^j |w^{(j)}(y)| \leq c_j, \end{cases}$$

for all $j \geq 0$ and some positive real numbers c_j . We call $w(y)$ an X -dyadic weight function.

We are interested in $\psi(y) = J_{2\kappa-1}(\theta \sqrt{y}) w(y)$ with $\theta > 0$, $\kappa \geq 1$ and $w(y)$ an X -dyadic weight function with $y^j w^{(j)}(y) \ll_j 1$.

4.8.1. For $\theta \sqrt{X} \ll 1$ we use the bound (4.21) for $J_{2\kappa-1}$, then it follows from repeating integration by parts that

$$\tilde{\psi}(-\sigma - it) \ll_{\sigma, A, \kappa} (\theta \sqrt{X})^{2\kappa-1} X^{-\sigma} (|t| + 1)^{-A}.$$

This bound with $A = j + 3\sigma + 3$ and the bound (4.25) for $G_\eta(s)$ yield, for $\sigma \geq -\frac{1}{2}$,

$$\begin{aligned} & x^j \Psi_\pm^{(j)}(x) \\ & \ll_{j,\sigma,\kappa} (\theta \sqrt{X})^{2\kappa-1} \int_{-\infty}^{\infty} (xX)^{-\sigma} ((|t|+1)(t^2+k^2))^{\sigma+\frac{1}{2}} (|t|+1)^{-3\sigma-3} dt \\ & \ll (\theta \sqrt{X})^{2\kappa-1} k \left(\frac{k^2}{xX} \right)^\sigma. \end{aligned}$$

If $xX \leq k^2(kX)^\varepsilon$, we choose $\sigma = 0$ and obtain

$$(4.27) \quad x^j \Psi_\pm^{(j)}(x) \ll_j (\theta \sqrt{X})^{2\kappa-1} k.$$

Otherwise it is negligible by choosing σ sufficiently large.

4.8.2. For $\theta \sqrt{X} \gg 1$ we use the expression (4.22) to write

$$\tilde{\psi}(-\sigma - it) = \sum_{\pm} \frac{1}{\sqrt{2\pi\theta}} \int_0^\infty e^{\pm i\theta \sqrt{y}} W_{2\kappa-1,\pm}(\theta \sqrt{y}) w(y) y^{-\sigma-it-\frac{1}{4}} \frac{dy}{y},$$

then repeating integration by parts and the bound (4.23) for $W_{\pm,2\kappa-1}$ yield

$$\tilde{\psi}(-\sigma - it) \ll_{\sigma,A,\kappa} \frac{1}{\sqrt{\theta} X^{\frac{1}{4}}} X^{-\sigma} \left(\frac{|t|}{\theta \sqrt{X}} + 1 \right)^{-A}.$$

This bound with $A = 3\sigma + 3$ and the bound (4.25) for $G_\eta(s)$ yield, for $\sigma \geq -\frac{1}{2}$,

$$\begin{aligned} & \Psi_\pm(x) \\ & \ll_{j,\sigma} \frac{1}{\sqrt{\theta} X^{\frac{1}{4}}} \int_{-\infty}^{\infty} (xX)^{-\sigma} ((|t|+1)(t^2+k^2))^{\sigma+\frac{1}{2}} \left(\frac{|t|}{\theta \sqrt{X}} + 1 \right)^{-3\sigma-3} dt \\ & \ll \theta \sqrt{X} k \left(\frac{(\theta \sqrt{X})^3 k^2}{xX} \right)^\sigma. \end{aligned}$$

If $xX \leq (\theta \sqrt{X})^3 k^2(kX)^\varepsilon$, we choose $\sigma = 0$ and obtain

$$(4.28) \quad \Psi_\pm(x) \ll_j \theta \sqrt{X} k.$$

Otherwise it is negligible by choosing σ sufficiently large.

4.9. A Wilton-type bound. We have the following Wilton-type bound involving conductor for $\text{Sym}^2 f$ ([4, Theorem 4.1]). This type of bound was first proved for symmetric square lifts of $\text{GL}(2, \mathbb{Z})$ -Maass forms in [15].

Proposition 4.4. *Let $X > 0$ and $w(x)$ be an X -dyadic weight function defined in (4.26). Then for any real number α ,*

$$(4.29) \quad \sum_{n=1}^{\infty} A_F(1, n) e(\alpha n) w(n) \ll_{\varepsilon, \{c_j\}_j} X^{\frac{3}{4}+\varepsilon} k^{\frac{1}{2}+\varepsilon}.$$

5. PROOF OF THEOREM 2.1

5.1. Amplified first moment average. Let $f \in S_k(1)$, $g_0 \in H_{2\kappa}^*(P)$, and assume that $\kappa \geq 2$ is fixed, $k > \kappa$ and P is prime. Then $Q := k^4 P^3$ is essentially the conductor of $\text{Sym}^2 f \otimes g$. We shall estimate the twisted first moment,

$$\mathfrak{F}_F(\ell) := \sum_{g \in H_{2\kappa}^*(P)} \omega_g^{-1} \lambda_g(\ell) L\left(\frac{1}{2}, \text{Sym}^2 f \otimes g\right)$$

when $(\ell, P) = 1$ and $\ell \leq 16L^4$ with L to be chosen. Subsequently, in the interest of simplifying notation, we shall write F for $\text{Sym}^2 f$.

Applying the approximate functional equation (4.11), we have for $Y > 0$

$$\begin{aligned} L\left(\frac{1}{2}, F \otimes g\right) &= \sum_{(d,P)=1} \sum_{n=1}^{\infty} \frac{\mu(d) A_F(1, n) \lambda_g(dn) \lambda_g(\ell)}{\sqrt{d^3 n}} V\left(\frac{d^3 n}{YP^{\frac{3}{2}}}\right) \\ &+ (-1)^{\kappa} \sqrt{P} \sum_{(d,P)=1} \sum_{n=1}^{\infty} \frac{\mu(d) A_F(1, n) \lambda_g(dnP) \lambda_g(\ell)}{\sqrt{d^3 n}} V\left(\frac{d^3 nY}{P^{\frac{3}{2}}}\right), \end{aligned}$$

where we have the applied multiplicative relation (4.2) which yields $\lambda_g(dn) \lambda_g(P) = \lambda_g(dnP)$. Therefore,

$$\mathfrak{F}_F(\ell) = S_1(\ell, Y) + (-1)^{\kappa} S_2(\ell, Y),$$

with

$$S_1(\ell) = S_1(\ell, Y) := \sum_{(d,P)=1} \frac{\mu(d)}{d^{\frac{3}{2}}} \sum_{n=1}^{\infty} \frac{A_F(1, n) \Delta_{2\kappa, P}^*(\ell, dn)}{\sqrt{n}} V\left(\frac{d^3 n}{YP^{\frac{3}{2}}}\right)$$

and

$$S_2(\ell) = S_2(\ell, Y) := \sqrt{P} \sum_{(d,P)=1} \frac{\mu(d)}{d^{\frac{3}{2}}} \sum_{n=1}^{\infty} \frac{A_F(1, n) \Delta_{2\kappa, P}^*(\ell, dnP)}{\sqrt{n}} V\left(\frac{d^3 nY}{P^{\frac{3}{2}}}\right).$$

It follows from the bound (4.12) for V that the contribution from $d^3 n > YQ^{\frac{1}{2}+\varepsilon}$ to $S_1(\ell)$ and that from $d^3 n > \frac{Q^{\frac{1}{2}+\varepsilon}}{Y}$ to $S_2(\ell)$ are negligible.

Furthermore, we shall make the following *a priori* assumption on L and Y ,

$$(5.1) \quad L^2 \sqrt{Y} Q^{\frac{1}{4}} \leq P$$

and see that our final choices of Y in (5.18) and L in (2.4) satisfy this assumption.

Remark 5.1. *The assumption (5.1) will not be used until the Wilton-type bound (Proposition 4.4) is applied to the final estimates in §5.5.1. One reason of making this assumption is so that the weight function after the integral transform in Voronoï (see §4.8.1) satisfies (4.26) the hypothesis in Proposition 4.4.*

5.2. Preparations for application of Petersson's trace formula.

5.2.1. *Treating $S_1(\ell)$.* Applying (4.20), the sum $S_1(\ell)$ is converted to

$$\begin{aligned} & \sum_{(d,P)=1} \mu(d) \left(\sum_{n=1}^{\infty} \frac{A_F(1,n)}{\sqrt{d^3 n}} V\left(\frac{d^3 n}{YP^{\frac{3}{2}}}\right) \Delta_{2\kappa,P}(\ell, dn) \right. \\ & - \sum_{n=1}^{\infty} \frac{A_F(1,n)}{\sqrt{d^3 n}} V\left(\frac{d^3 n}{YP^{\frac{3}{2}}}\right) \cdot \frac{1}{Pv((n,P))} \sum_{j=0}^{\infty} P^{-j} \Delta_{2\kappa,P}(\ell P^{2j}, dn) \\ & \left. + \sum_{n \equiv 0 \pmod{P^2}} \frac{A_F(1,n)}{\sqrt{d^3 n}} V\left(\frac{d^3 n}{YP^{\frac{3}{2}}}\right) \cdot \frac{1}{P+1} \sum_{j=0}^{\infty} P^{-j} \Delta_{2\kappa,P}(\ell P^{2j}, dn P^{-2}) \right). \end{aligned}$$

By trivial estimates using Deligne's bound, the bound (4.12) for V and the bound (4.17) for $\Delta_{2\kappa,P}$, the sum of the last two terms is bounded by

$$\begin{aligned} & \sum_{(d,P)=1} \left(\sum_{d^3 n \leq YQ^{\frac{1}{2}+\varepsilon}} \frac{(\ell dn P)^{\varepsilon}}{\sqrt{d^3 n P}} + \sum_{\substack{n \equiv 0 \pmod{P^2} \\ d^3 n \leq YQ^{\frac{1}{2}+\varepsilon}}} \frac{(\ell dn P)^{\varepsilon}}{\sqrt{d^3 n (P+1)}} \right) \\ & \ll \frac{\sqrt{Y} Q^{\frac{1}{4}+\varepsilon}}{P} \sum_{(d,P)=1} \frac{1}{d^{3-\varepsilon}} \ll \frac{\sqrt{Y} Q^{\frac{1}{4}+\varepsilon}}{P}. \end{aligned}$$

Therefore,

$$S_1(\ell) = T_1(\ell) + O\left(\frac{\sqrt{Y} Q^{\frac{1}{4}+\varepsilon}}{P}\right),$$

with

$$(5.2) \quad T_1(\ell) := \sum_{(d,P)=1} \mu(d) \sum_{n=1}^{\infty} \frac{A_F(1,n)}{\sqrt{d^3 n}} V\left(\frac{d^3 n}{YP^{\frac{3}{2}}}\right) \Delta_{2\kappa,P}(\ell, dn).$$

5.2.2. *Treating $S_2(\ell)$.* Similarly, after applying (4.20), the sum $S_2(\ell)$ turns into

$$\begin{aligned} & \sum_{(d,P)=1} \mu(d) \left(\sqrt{P} \sum_{n=1}^{\infty} \frac{A_F(1,n)}{\sqrt{d^3 n}} V\left(\frac{d^3 n Y}{P^{\frac{3}{2}}}\right) \Delta_{2\kappa,P}(\ell, dn P) \right. \\ & - \sqrt{P} \sum_{n=1}^{\infty} \frac{A_F(1,n)}{\sqrt{d^3 n}} V\left(\frac{d^3 n Y}{P^{\frac{3}{2}}}\right) \cdot \frac{1}{P(P+1)} \sum_{j=0}^{\infty} P^{-j} \Delta_{2\kappa,P}(P^{2j} \ell, dn P) \\ & \left. + \sqrt{P} \sum_{n \equiv 0 \pmod{P}} \frac{A_F(1,n)}{\sqrt{d^3 n}} V\left(\frac{d^3 n Y}{P^{\frac{3}{2}}}\right) \cdot \frac{1}{P+1} \sum_{j=0}^{\infty} P^{-j} \Delta_{2\kappa,P}(P^{2j} \ell, dn P^{-1}) \right). \end{aligned}$$

By trivial estimates we have

$$S_2(\ell) = T_2(\ell) + O\left(\frac{Q^{\frac{1}{4}+\varepsilon}}{\sqrt{Y}P^{\frac{3}{2}}}\right),$$

with

$$(5.3) \quad T_2(\ell) := \sqrt{P} \sum_{(d,P)=1} \mu(d) \sum_{n=1}^{\infty} \frac{A_F(1,n)}{\sqrt{d^3 n}} V\left(\frac{d^3 n Y}{P^{\frac{3}{2}}}\right) \Delta_{2\kappa,P}(\ell, dnP).$$

5.3. Application of Petersson's formula.

5.3.1. *Treating $T_1(\ell)$.* Applying Petersson's formula (4.18), $T_1(\ell)$ defined in (5.2) is equal to

$$\begin{aligned} & \sum_{(d,P)=1} \mu(d) \sum_{n=1}^{\infty} \frac{A_F(1,n)}{\sqrt{d^3 n}} V\left(\frac{d^3 n}{YP^{\frac{3}{2}}}\right) \\ & \times \left(\delta(\ell, dn) + (-1)^\kappa 2\pi \sum_{\substack{c>0 \\ c \equiv 0 \pmod{P}}} \frac{S(\ell, dn; c)}{c} J_{2\kappa-1}\left(\frac{4\pi\sqrt{\ell dn}}{c}\right) \right). \end{aligned}$$

Trivial estimates using Deligne's bound and the bound (4.14) for V yields the following estimate on the diagonal term

$$\sum_{\substack{(d,P)=1 \\ d|\ell}} \frac{A_F(1, \ell/d)}{\sqrt{\ell d^2}} V\left(\frac{\ell d^2}{YP^{\frac{3}{2}}}\right) \ll \frac{Q^\varepsilon}{\sqrt{\ell}}.$$

Consider now the size of the off-diagonal terms

$$\sum_{(d,P)=1} \mu(d) \sum_{n=1}^{\infty} \frac{A_F(1,n)}{\sqrt{d^3 n}} \sum_{\substack{c>0 \\ c \equiv 0 \pmod{P}}} \frac{S(\ell, dn; c)}{c} J_{2\kappa-1}\left(\frac{4\pi\sqrt{\ell dn}}{c}\right) V\left(\frac{d^3 n}{YP^{\frac{3}{2}}}\right).$$

We wish to apply the Voronoï summation formula on the sum over n . To this end, we subdivide the sum over n by a smooth dyadic partition of unity for V , open the Kloosterman sum, and obtain the following sum up to a negligible error term due to (4.12),

$$T_1^o(\ell) := \sum_{X \leq YQ^{\frac{1}{2}+\varepsilon}} \sum_{(d,P)=1} \mu(d) T_1^o(X, d, \ell),$$

where X is of the form $2^{\frac{j}{2}}$ with $j \geq -1$,

$$T_1^o(X, d, \ell) := \sum_{\substack{c>0 \\ c \equiv 0 \pmod{P}}} \frac{1}{c} \sum_{a \pmod{c}}^* e\left(\frac{\ell a}{c}\right) \sum_{n=1}^{\infty} A_F(1,n) e\left(\frac{dn\bar{a}}{c}\right) \psi_1(n; c, X, d, \ell),$$

in which

$$\begin{aligned}\psi_1(y; c, X, d, \ell) &:= J_{2\kappa-1} \left(\frac{4\pi \sqrt{\ell d y}}{c} \right) w_1(y; X, d), \\ w_1(y; X, d) &:= \frac{1}{\sqrt{d^3 y}} V \left(\frac{d^3 y}{Y P^{\frac{3}{2}}} \right) h \left(\frac{d^3 y}{X} \right),\end{aligned}$$

with $h(y)$ some smooth function supported on $[1, 2]$ satisfying $h^{(j)}(y) \ll_j 1$. In view of the range of X , the bound (4.14) for V implies that $w_1(y; X, d)$ is an X -dyadic weight function with bounds

$$(5.4) \quad x^j w_1^{(j)}(y; X, d) \ll_{j, \varepsilon, l} \frac{Q^\varepsilon}{\sqrt{X}}.$$

Moreover, we pull out the greatest common divisor (c, d) by writing

$$T_1^o(X, d, \ell) = \sum_{d_1 d_2 = d} \sum_{\substack{(c, d_2) = 1 \\ c \equiv 0 \pmod{P}}} T_1^o(c, X, d_1, d_2, \ell),$$

with

$$(5.5) \quad \begin{aligned} T_1^o(c, X, d_1, d_2, \ell) &= \frac{1}{c d_1} \sum_{a \pmod{c d_1 P}}^* e \left(\frac{\ell a}{c d_1} \right) \\ &\quad \sum_{n=1}^{\infty} A_F(1, n) e \left(\frac{d_2 n \bar{a}}{c} \right) \psi_1(n; c d_1, X, d_1 d_2, \ell), \end{aligned}$$

5.3.2. *Treating $T_2(\ell)$.* The diagonal term coming from Petersson's formula applied to $T_2(\ell)$ vanishes since $(\ell, P) = 1$. The off-diagonal sum is of size

$$\sum_{(d, P)=1} \mu(d) \sum_{n=1}^{\infty} \frac{A_F(1, n)}{\sqrt{d^3 n}} V \left(\frac{d^3 n}{Y P^{\frac{3}{2}}} \right) \sum_{c=1}^{\infty} \frac{S(\ell, d n P; c P)}{c \sqrt{P}} J_{2\kappa-1} \left(\frac{4\pi \sqrt{\ell d n}}{c \sqrt{P}} \right).$$

Since $S(\ell, d n P; c P) = 0$ when $P|c$, we may impose the condition $(c, P) = 1$ to the sum over c . Under this condition

$$S(\ell, d n P; c P) = S(\ell \bar{c}, 0; P) S(\ell \bar{P}, d n; c) = -S(\ell \bar{P}, d n; c).$$

Inserting this above and reordering our sums we obtain

$$\sum_{(d, P)=1} \mu(d) \sum_{(c, P)=1} \frac{1}{c \sqrt{P}} \sum_{n=1}^{\infty} \frac{A_F(1, n)}{\sqrt{d^3 n}} S(\ell \bar{P}, d n; c) J_{2\kappa-1} \left(\frac{4\pi \sqrt{\ell d n}}{c \sqrt{P}} \right) V \left(\frac{d^3 n}{Y P^{\frac{3}{2}}} \right).$$

With the same treatment as in §5.3.1, one is left to study the sum

$$T_2^o(\ell) := \sum_{X \leq Q^{\frac{1}{2} + \varepsilon} / Y} \sum_{(d, P)=1} \mu(d) T_2^o(X, d, \ell)$$

with

$$T_2^o(X, d, \ell) := \sum_{d_1 d_2 = d} \sum_{(c, P d_2) = 1} T_2^o(c, X, d_1, d_2, \ell),$$

and

$$T_2^o(c, X, d_1, d_2, \ell) := \frac{1}{cd_1 \sqrt{P}} \sum_{a \pmod{cd_1}}^* e\left(\frac{\ell \bar{P} a}{cd_1}\right) \sum_{n=1}^{\infty} A_F(1, n) e\left(\frac{d_2 n \bar{a}}{c}\right) \psi_2(n; cd_1, X, d_1 d_2, \ell),$$

where

$$\psi_2(y; c, X, d, \ell) := J_{2K-1}\left(\frac{4\pi \sqrt{\ell d y}}{c \sqrt{P}}\right) w_2(y; X, d)$$

and

$$w_2(y; X, d) := \frac{1}{\sqrt{d^3 y}} V\left(\frac{d^3 y}{Y P^{\frac{3}{2}}}\right) h\left(\frac{d^3 y}{X}\right).$$

Again, $w_2(y; X, d)$ is an X -dyadic weight function with bounds

$$(5.6) \quad x^j w_2^{(j)}(y; X, d) \ll_{j, \varepsilon, l} \frac{Q^\varepsilon}{\sqrt{X}}.$$

5.4. Application of Voronoi's formula.

5.4.1. *Treating $T_1^o(c, X, d_1, d_2, \ell)$.* Applying Voronoi's formula to the innermost sum in (5.5), $T_1^o(c, X, d_1, d_2, \ell)$ is converted to the sum

$$T_{1,+}^o(c, X, d_1, d_2, \ell) + T_{1,-}^o(c, X, d_1, d_2, \ell),$$

where

$$T_{1,\pm}^o(c, X, d_1, d_2, \ell) := \frac{1}{d_1} \sum_{a \pmod{cd_1}}^* e\left(\frac{\ell a}{cd_1}\right) \sum_{n_1 c_1 = c} \sum_{n_2 = 1}^{\infty} \frac{A_F(n_2, n_1)}{n_1 n_2} S(a \bar{d}_2, \pm n_2; c_1) \Psi_{1,\pm}\left(\frac{n_2 n_1^2}{c^3}; cd_1, X, d_1 d_2, \ell\right),$$

and $\Psi_{1,\pm}(x; c, X, d, \ell)$ is the integral transform of $\psi_{1,\pm}(y; c, X, d, \ell)$ defined as in §4.6. Opening the Kloosterman sum and changing the order of summation in $T_{1,\pm}^o(c, X, d_1, d_2, \ell)$ above, we arrive at

$$T_{1,\pm}^o(c, X, d_1, d_2, \ell) = \frac{1}{d_1} \sum_{n_1 c_1 = c} \frac{1}{n_1} \sum_{b \pmod{c_1}}^* \sum_{a \pmod{cd_1}}^* e\left(\frac{a(b d_1 n_1 \bar{d}_2 + \ell)}{cd_1}\right) \sum_{n_2 = 1}^{\infty} \frac{A_F(n_2, n_1)}{n_2} e\left(\pm \frac{n_2 \bar{b}}{c_1}\right) \Psi_{1,\pm}\left(\frac{n_2 n_1^2}{c^3}; cd_1, X, d_1 d_2, \ell\right).$$

By Möbius inversion,

$$\sum_{a \pmod{cd_1}}^* e\left(\frac{a(bd_1n_1\bar{d}_2 + \ell)}{cd_1}\right) = \sum_{c_2|cd_1} \mu\left(\frac{cd_1}{c_2}\right) \sum_{a \pmod{c_2}} e\left(\frac{a(bd_1n_1\bar{d}_2 + \ell)}{c_2}\right).$$

The inner sum produces a congruence condition $bd_1n_1 \equiv -d_2\ell \pmod{c_2}$, which forces $(c_2, d_1n_1) | (c_2, \ell)$ and hence $c_2 | c_1(d_1n_1, \ell)$. Therefore, $T_{1,\pm}^o(X, d_1, d_2, \ell)$ becomes

$$\begin{aligned} & \frac{1}{d_1} \sum_{n_1c_1=c} \sum_{c_2|c_1(d_1n_1, \ell)} \mu\left(\frac{cd_1}{c_2}\right) \frac{c_2}{n_1} \\ & \sum_{\substack{b \pmod{c_1} \\ b \equiv -\ell d_2 \bar{d}_1 n_1 \pmod{c_2}}}^* \sum_{n_2=1}^{\infty} e\left(\pm \frac{n_2 \bar{b}}{c_1}\right) \frac{A_F(n_2, n_1)}{n_2} \Psi_{1,\pm}\left(\frac{n_2 n_1^2}{c^3}; cd_1, X, d_1 d_2, \ell\right). \end{aligned}$$

Finally, we apply the Hecke relation (4.5),

(5.7)

$$\begin{aligned} & T_{1,\pm}^o(c, X, d_1, d_2, \ell) \\ &= \frac{1}{d_1} \sum_{n_1c_1=c} \sum_{c_2|c_1(d_1n_1, \ell)} \sum_{n_3|n_1} \mu\left(\frac{cd_1}{c_2}\right) \mu(n_3) \frac{c_2}{n_1 n_3} A_F(1, n_1/n_3) \\ & \sum_{\substack{b \pmod{c_1} \\ b \equiv -\ell d_2 \bar{d}_1 n_1 \pmod{c_2}}}^* \sum_{n=1}^{\infty} \frac{A_F(n, 1)}{n} e\left(\pm \frac{nn_3 \bar{b}}{c_1}\right) \Psi_{1,\pm}\left(\frac{nn_3 n_1^2}{c^3}; cd_1, X, d_1 d_2, \ell\right). \end{aligned}$$

5.4.2. *Treating $T_2^o(c, X, d_1, d_2, \ell)$.* Following the same line of arguments as in §5.4.1, we have

$$T_2^o(c, X, d_1, d_2, \ell) = T_{2,+}^o(c, X, d_1, d_2, \ell) + T_{2,-}^o(c, X, d_1, d_2, \ell)$$

with

(5.8)

$$\begin{aligned} & T_{2,\pm}^o(c, X; d_1, d_2, \ell) \\ &= \frac{1}{\sqrt{P}d_1} \sum_{n_1c_1=c} \sum_{c_2|c_1(d_1n_1, \ell)} \sum_{n_3|n_1} \mu\left(\frac{cd_1}{c_2}\right) \mu(n_3) \frac{c_2}{n_1 n_3} A_F(1, n_1/n_3) \\ & \sum_{\substack{b \pmod{c_1} \\ b \equiv -\ell d_2 \bar{d}_1 n_1 P \pmod{c_2}}}^* \sum_{n=1}^{\infty} \frac{A_F(n, 1)}{n} e\left(\pm \frac{nn_3 \bar{b}}{c_1}\right) \Psi_{2,\pm}\left(\frac{nn_3 n_1^2}{c^3}; cd_1, X, d_1 d_2, \ell\right). \end{aligned}$$

5.5. Final estimates. We restrict ourselves to the partial sums of (5.7) and (5.8) in which $d = n_1 = 1$ to simplify the complicated notation; the general cases may be treated in the same way.

5.5.1. Suppose $X \leq YQ^{\frac{1}{2}+\varepsilon}$ and $c \geq P$ with $c \equiv 0 \pmod{P}$. Let

$$\psi_1(y; c, X, \ell) := \psi_1(y; c, X, 1, \ell) = J_{2\kappa-1} \left(\frac{4\pi\sqrt{\ell y}}{c} \right) w_1(y; X),$$

with

$$w_1(y; X) := w_1(y; X, 1) = \frac{1}{\sqrt{y}} V \left(\frac{yY}{P^{\frac{3}{2}}} \right) h \left(\frac{y}{X} \right).$$

We consider the sum

$$(5.9) \quad T_{1,\pm}^o(c, X, \ell) := \sum_{\substack{c_2|c \\ (c_2, \ell)=1}} \mu \left(\frac{c}{c_2} \right) c_2 \sum_{\substack{b \pmod{c} \\ b \equiv -\ell \pmod{c_2}}}^* \sum_{n=1}^{\infty} \frac{A_F(n, 1)}{n} e \left(\pm \frac{n\bar{b}}{c} \right) \Psi_{1,\pm} \left(\frac{n}{c^3}; c, X, \ell \right).$$

Then summing all $T_{1,\pm}^o(c, X, \ell)$ over c and \pm yields the partial sum of $T_1^o(X, 1, \ell)$ with $n_1 = 1$.

By our assumption (5.1) we have $\frac{4\pi\sqrt{\ell X}}{c} \ll \frac{L^2\sqrt{X}}{P} \leq 1$, so we are in the situation of §4.8.1 with $\theta = \frac{4\pi\sqrt{\ell}}{c}$ and $w(y) = w_1(y; X) \frac{\sqrt{X}}{Q^\varepsilon}$ in view of (5.4). According to (4.27), for $xX \leq k^2 Q^\varepsilon$ we have the following bounds

$$(5.10) \quad x^j \Psi_{1,\pm}^{(j)}(x; c, X, \ell) \ll_{j,\varepsilon} \frac{kQ^\varepsilon}{\sqrt{X}} \left(\frac{\sqrt{\ell X}}{c} \right)^{2\kappa-1}.$$

Otherwise, the bounds are arbitrarily small.

Next we wish to use the Wilton-type bound in the form (4.29). For this we apply a dyadic partition of unity and convert the innermost sum over n in (5.9) into

$$(5.11) \quad \sum_Z \sum_{n=1}^{\infty} A_F(n, 1) e \left(\pm \frac{n\bar{b}}{c} \right) w_{1,\pm}(n; c, X, Z, \ell),$$

where $Z = 2^{\frac{j}{2}}$ with $j \geq -1$ and

$$w_{1,\pm}(x; c, X, Z, \ell) := \frac{1}{x} \Psi_{1,\pm} \left(\frac{x}{c^3}; c, X, \ell \right) h \left(\frac{x}{Z} \right).$$

The contribution from $Z > \frac{c^3 k^2 Q^\varepsilon}{X}$ is negligible. Otherwise, the bound (5.10) implies that $w_{1,\pm}(x; c, X, Z, \ell)$ is a Z -dyadic weight function with

$$x^j w_{1,\pm}^{(j)}(x; c, X, Z, \ell) \ll_{j,\varepsilon} \frac{kQ^\varepsilon}{\sqrt{XZ}} \left(\frac{\sqrt{\ell X}}{c} \right)^{2\kappa-1}.$$

Applying the Wilton-type bound (4.29) to the inner sum in (5.11) gives

$$\begin{aligned} T_{1,\pm}^o(c, X, \ell) &\ll \sum_{\substack{c_2|c \\ (c_2, \ell)=1}} c \sum_{Z \leq c^3 k^2 Q^\varepsilon / X} \frac{k^{\frac{3}{2}} Q^\varepsilon}{\sqrt{X} Z^{\frac{1}{4}}} \left(\frac{\sqrt{\ell X}}{c} \right)^{2\kappa-1} \\ &\ll \frac{c^{\frac{1}{4}+\varepsilon} k Q^\varepsilon}{X^{\frac{1}{4}}} \left(\frac{\sqrt{\ell X}}{c} \right)^{2\kappa-1}. \end{aligned}$$

Therefore,

(5.12)

$$\sum_{\substack{c \geq P \\ c \equiv 0 \pmod{P}}} T_{1,\pm}^o(c, X, \ell) \ll \frac{k Q^\varepsilon}{X^{\frac{1}{4}}} \sum_{\substack{c \geq P \\ c \equiv 0 \pmod{P}}} c^{\frac{1}{4}+\varepsilon} \left(\frac{\sqrt{\ell X}}{c} \right)^{2\kappa-1} \ll \frac{\ell^{\frac{5}{8}} X^{\frac{3}{8}} k Q^\varepsilon}{P},$$

where it should be noted that the assumption $\kappa \geq 2$ is required to guarantee the convergence of the c -sum.

Finally, summing over X , the partial sum of $T_1^o(\ell)$ in which $d = n_1 = 1$ is bounded by

$$\sum_{X \leq Y Q^{\frac{1}{2}+\varepsilon}} \frac{\ell^{\frac{5}{8}} X^{\frac{3}{8}} k Q^\varepsilon}{P} \ll \frac{\ell^{\frac{5}{8}} Y^{\frac{3}{8}} k Q^{\frac{3}{16}+\varepsilon}}{P}.$$

5.5.2. Suppose $X \leq \frac{Q^{\frac{1}{2}+\varepsilon}}{Y}$ and $(c, P) = 1$. Let

$$\psi_2(y; c, X, \ell) := \psi_2(y; c, X, 1, \ell) = J_{2\kappa-1} \left(\frac{4\pi \sqrt{\ell y}}{c \sqrt{P}} \right) w_2(y; X),$$

with

$$w_2(y; X) := w_2(y; X, 1) = \frac{1}{\sqrt{y}} V \left(\frac{y}{Y P^{\frac{3}{2}}} \right) h \left(\frac{y}{X} \right).$$

Consider the sum

$$\begin{aligned} T_{2,\pm}^o(c, X, \ell) &:= \frac{1}{\sqrt{P}} \sum_{\substack{c_2|c \\ (c_2, \ell)=1}} \mu \left(\frac{c}{c_2} \right) c_2 \\ (5.13) \quad &\sum_{\substack{b \pmod{c} \\ b \equiv -\ell \bar{P} \pmod{c_2}}}^* \sum_{n=1}^{\infty} \frac{A_F(n, 1)}{n} e \left(\pm \frac{n \bar{b}}{c} \right) \Psi_{2,\pm} \left(\frac{n}{c^3}; c, X, \ell \right). \end{aligned}$$

All $T_{2,\pm}^o(c, X, \ell)$ constitute the partial sum of $T_2^o(X, 1, \ell)$ with $n_1 = 1$.

We shall apply arguments similar to those in §5.5.1 except that, instead of using the Wilton-type bound, Deligne's bound is trivially applied to each individual $A_F(n, 1)$.

When $c \geq \sqrt{\frac{\ell X}{P}}$, the analysis of §4.8.1 implies that we may truncate the sum over n in (5.13) at $\frac{c^3 k^2 Q^\varepsilon}{X}$ with the difference of a negligible error, and for $n \leq \frac{c^3 k^2 Q^\varepsilon}{X}$ we have

$$\Psi_{2,\pm}\left(\frac{n}{c^3}; c, X, \ell\right) \ll \frac{kQ^\varepsilon}{\sqrt{X}} \left(\frac{\sqrt{\ell X}}{c\sqrt{P}}\right)^{2\kappa-1}.$$

Thus trivial estimates yield

$$\begin{aligned} T_{2,\pm}^o(c, X, \ell) &\ll \frac{1}{\sqrt{P}} \sum_{c_2|c} c \sum_{n \leq c^3 k^2 Q^\varepsilon / X} \frac{kQ^\varepsilon}{\sqrt{X}n} \left(\frac{\sqrt{\ell X}}{c\sqrt{P}}\right)^{2\kappa-1} \\ (5.14) \quad &\ll \frac{c^{1+\varepsilon} kQ^\varepsilon}{\sqrt{XP}} \left(\frac{\sqrt{\ell X}}{c\sqrt{P}}\right)^{2\kappa-1}. \end{aligned}$$

When $c < \sqrt{\frac{\ell X}{P}}$, we are in the situation of §4.8.2. It follows from (4.28) that if n does not exceed $\frac{(\sqrt{\ell X}/(c\sqrt{P}))^3 c^3 k^2 Q^\varepsilon}{X}$ then we have the bound

$$\Psi_{2,\pm}\left(\frac{n}{c^3}; c, X, \ell\right) \ll \frac{kQ^\varepsilon}{\sqrt{X}} \frac{\sqrt{\ell X}}{c\sqrt{P}}.$$

Otherwise, we have a negligible contribution. Therefore,

$$\begin{aligned} T_{2,\pm}^o(c, X, \ell) &\ll \frac{1}{\sqrt{P}} \sum_{c_2|c} c \sum_{n \leq (\sqrt{\ell X}/(c\sqrt{P}))^3 c^3 k^2 Q^\varepsilon / X} \frac{kQ^\varepsilon}{\sqrt{X}n} \frac{\sqrt{\ell X}}{c\sqrt{P}} \\ (5.15) \quad &\ll \frac{c^{1+\varepsilon} kQ^\varepsilon}{\sqrt{XP}} \frac{\sqrt{\ell X}}{c\sqrt{P}}. \end{aligned}$$

Combining (5.14, 5.15) we have

$$\begin{aligned} (5.16) \quad &\sum_{(c,P)=1} T_{1,\pm}^o(c, X, \ell) \\ &\ll \frac{kQ^\varepsilon}{\sqrt{XP}} \left(\sum_{\substack{c \geq \sqrt{\ell X/P} \\ (c,P)=1}} c^{1+\varepsilon} \left(\frac{\sqrt{\ell X}}{c\sqrt{P}}\right)^{2\kappa-1} + \sum_{\substack{c < \sqrt{\ell X/P} \\ (c,P)=1}} c^{1+\varepsilon} \frac{\sqrt{\ell X}}{c\sqrt{P}} \right) \ll \frac{\ell \sqrt{X} kQ^\varepsilon}{P^{\frac{3}{2}}}. \end{aligned}$$

Again, the assumption $\kappa \geq 2$ guarantees the convergence of this c -sum.

We conclude with the following bound for the partial sum of $T_2^o(\ell)$ where $d = n_1 = 1$,

$$\sum_{X \leq Q^{\frac{1}{2}+\varepsilon}/Y} \frac{\ell \sqrt{X} kQ^\varepsilon}{P^{\frac{3}{2}}} \ll \frac{\ell k Q^{\frac{1}{4}+\varepsilon}}{\sqrt{Y} P^{\frac{3}{2}}}.$$

5.6. Conclusion. In conclusion, summing all contributions in the above arguments, we have the following bound for the twisted first moment

$$\begin{aligned}\mathfrak{F}_F(\ell) &= \sum_{g \in H_{2k}^*(P)} \omega_g^{-1} \lambda_g(\ell) L\left(\frac{1}{2}, F \otimes g\right) \\ &\ll \left(\frac{\sqrt{Y}Q^{\frac{1}{4}}}{P} + \frac{Q^{\frac{1}{4}}}{\sqrt{Y}P^{\frac{3}{2}}} + \frac{1}{\sqrt{\ell}} + \frac{\ell^{\frac{5}{8}}Y^{\frac{3}{8}}kQ^{\frac{3}{16}}}{P} + \frac{\ell kQ^{\frac{1}{4}}}{\sqrt{Y}P^{\frac{3}{2}}} \right) Q^\varepsilon,\end{aligned}$$

where $\ell \leq 16L^4$ is co-prime with P and with L and Y satisfying our assumption (5.1). Thus,

$$\begin{aligned}(5.17) \quad \mathfrak{F}_F(\ell) &\ll \left(\frac{1}{\sqrt{\ell}} + \frac{\sqrt{Y}Q^{\frac{1}{4}}}{P} + \frac{Q^{\frac{1}{4}}}{\sqrt{Y}P^{\frac{3}{2}}} + \frac{L^{\frac{5}{2}}Y^{\frac{3}{8}}kQ^{\frac{3}{16}}}{P} + \frac{L^4kQ^{\frac{1}{4}}}{\sqrt{Y}P^{\frac{3}{2}}} \right) Q^\varepsilon \\ &\ll \left(\frac{1}{\sqrt{\ell}} + \frac{\sqrt{Y}Q^{\frac{1}{4}}}{P} + \frac{L^{\frac{5}{2}}Y^{\frac{3}{8}}kQ^{\frac{3}{16}}}{P} + \frac{L^4kQ^{\frac{1}{4}}}{\sqrt{Y}P^{\frac{3}{2}}} \right) Q^\varepsilon \\ &\ll \left(\frac{1}{\sqrt{\ell}} + \frac{L^{\frac{5}{2}}Y^{\frac{3}{8}}kQ^{\frac{3}{16}}}{P} + \frac{L^4kQ^{\frac{1}{4}}}{\sqrt{Y}P^{\frac{3}{2}}} \right) Q^\varepsilon,\end{aligned}$$

where the last line follows from our assumption $L^2\sqrt{Y}Q^{\frac{1}{4}} \leq P$ in (5.1). We achieve an optimal bound by choosing

$$(5.18) \quad Y = L^{\frac{12}{7}}k^{\frac{2}{7}}P^{-\frac{5}{14}}.$$

Inserting this value for Y in (5.17) we get Theorem 2.1.

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